Bisection-closed families, tournaments, and symmetric designs

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Intersecting families of sets

A fractional variant

Definition (Balachandran–Mathew–Mishra 2019)

A family $\mathcal{F}$ of subsets of $[n]$ is bisection-closed if, for all $A, B \in \mathcal{F}$, $A \neq B$, we have

$$|A \cap B| \in \{\tfrac{1}{2}|A|, \tfrac{1}{2}|B|\}.$$

Example (Sunflower family)

Let $\mathcal{F}_s := \{12, 13, \ldots, 1n, 1234, 1256, \ldots, 12(n-1)n\}$.
Then, $\mathcal{F}_s$ is bisection-closed, and $|\mathcal{F}_s| = \frac{3n}{2} - 2$.

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Example (Hadamard family)

Let H be an $m \times m$ Hadamard matrix in normal form, and let J be the $m \times m$ all-ones matrix. Let $A_1, \ldots, A_{3m}$ be the rows of

$$\begin{bmatrix} H & H \\ H & -H \\ H & -J \end{bmatrix},$$

viewed as the $\{\pm 1\}$-incidence vectors of subsets of $[2m]$.

Then, $\mathcal{F}_H := \{A_i : i \in [3m] \setminus \{1, 2m+1\}\}$ is a bisection-closed family. Writing $2m = n$, we have $|\mathcal{F}_H| = \frac{3n}{2} - 2$.

Are these families extremal?

Even a linear upper bound is not known!

Theorem (Balachandran–Mathew–Mishra 2019)

If $\mathcal{F}$ is a bisection-closed family over $[n]$, then

$$|\mathcal{F}| \leq O(n \log n).$$

Conjecture (Balachandran–Mathew–Mishra 2019)

There is a constant $c > 0$ such that any bisection-closed family over $[n]$ has size at most cn.

A linear algebraic reformulation

Let $\mathcal{F}$ be a bisection-closed family over $[n]$ of size m.
Let $X_{m\times n}$ be the $\{\pm 1\}$-incidence matrix for $\mathcal{F}$. One can check:

$$XX^T(A,A) = n,$$

$$XX^T(A,B) = \begin{cases} n - 2|A|, & \text{if } |A\cap B| = \theta|B|; \\ n - 2|B|, & \text{if } |A\cap B| = \theta|A|. \end{cases}$$

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So, consider $M = \frac{1}{2}(nJ - XX^T)$. Since $\text{rank}(M) \leq n+1$,

$$\text{rank}(M) \geq \Omega(m) \implies |\mathcal{F}| \leq O(n).$$

Generalize!

Let $\mathbb{F}$ be a field, $a_1, \ldots, a_m \in \mathbb{F}^*$.

Let $\mathrm{Sym}(a_1, \ldots, a_m)$ denote the collection of all the $m \times m$ symmetric matrices over $\mathbb{F}$ with zero diagonal and (i, j)th entry either a_i or a_j, for all $i < j$.

Question

Is there an absolute constant $c > 0$ *such that* $\mathrm{rank}(M) \geq cm$ *for all* $M \in \mathrm{Sym}(a_1, \ldots, a_m)$*?*

A connection with tournaments

A tournament over $[m]$ is an orientation of the edges of the complete graph K_m.

Given $a_1, \ldots, a_m \in \mathbb{F}^*$, for each tournament T over $[m]$ we get a matrix $M_T \in \mathrm{Sym}(a_1, \ldots, a_m)$ by setting (for $i < j$):

$$M_T(i,j) = a_i \text{ if } i \to j \text{ in } T,$$
$$M_T(i,j) = a_j \text{ if } j \to i \text{ in } T.$$

Conversely, given a matrix $M \in \mathrm{Sym}(a_1, \ldots, a_m)$, we can define an associated tournament over $[m]$ (in possibly more than one way).

Matrices associated to *random* tournaments have high rank

Using McDiarmid's concentration inequality, we can show that a matrix associated to a uniformly random tournament has "high" rank with high probability.

Theorem (Balachandran–Bhattacharya–S. 2023)

Let $\mathrm{char}(\mathbb{F}) \neq 2$*, and let* $(a_k)_{k\geq 1}$ *be a sequence in* $\mathbb{F}^*$*. Let* T *be a uniformly random tournament over* $[n]$*, and* $M_T \in \mathrm{Sym}(a_1, \ldots, a_m)$ *be the associated matrix. Then, with high probability we have*

$$\mathrm{rank}(M_T) \geq \left(\frac{1}{2} - o(1)\right) m.$$

Families with only two distinct set sizes

The sunflower family $\mathcal{F}_s$ and Hadamard family $\mathcal{F}_H$ have sets of only two distinct sizes.

A trivial upper bound on the size of any such family over $[n]$ is $2n$, but the best known constructions give a bound of $\approx \frac{3n}{2}$.

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Can this bound be improved? Or, in the linear algebraic language:

Question

When can we improve upon the trivial bound $\mathrm{rank}(M) \geq m/2$ *for* $M \in \mathrm{Sym}(\alpha, \ldots, \alpha, \beta, \ldots, \beta)$*?*

Multiplicity of eigenvalues vs. ranks of matrices

Proposition (Balachandran–S. 2024)

Let $M \in \mathrm{Sym}(\alpha, \ldots, \alpha, \beta, \ldots, \beta)$ *be an* $m \times m$ *matrix. Let* $\mu \in \mathbb{C}$ *be given by*

$$\mu^2 = \frac{\alpha\beta}{(\alpha - \beta)^2}.$$

There is an associated bipartite graph G_M *such that, if* ν *is the multiplicity of* μ *as an eigenvalue of* G_M*, then*

$$|\mathrm{rank}(M) - (m - \nu)| \leq 2.$$

In particular, if M has "low" rank, then the bipartite graph G_M has an eigenvalue μ with "high" multiplicity.

Symmetric designs

To search for bipartite graphs with eigenvalues of high multiplicity, we turn to symmetric designs:

Definition

A symmetric 2-(v, k, λ) design Δ is a collection of k-subsets of $[v]$ such that every pair of elements in v belongs to exactly λ sets in the collection, and $|\Delta| = v$.

Any symmetric 2-(v, k, λ) design Δ has an associated bipartite point-block incidence graph G_Δ, which has spectrum

$$\{v, (\sqrt{k-\lambda})^{(v-1)}, (-\sqrt{k-\lambda})^{(v-1)}, -v\}.$$

Low-rank matrices in Sym$(\alpha, \dots, \alpha, \beta, \dots, \beta)$

Using the complete bipartite graph minus a matching, we show that we **cannot** improve upon the trivial bound for real matrices:

Theorem (Balachandran–S. 2024)

Let $\alpha = 1$ *and* $\beta = (3 + \sqrt{5})/2$. *For every* $m \in \mathbb{N}$, *there is an* $m \times m$ *matrix* $M \in \mathrm{Sym}(\alpha, \dots, \alpha, \beta, \dots, \beta)$ *with* $\mathrm{rank}(M) \leq \frac{m}{2} + 3$.

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Using the 2-$(4n-1, 2n-1, n-1)$ designs, called *Hadamard designs*, we show that we **cannot** improve upon the trivial bound for rational or integral matrices as well:

Theorem (Balachandran–S. 2024)

For each $\varepsilon > 0$, *there exists* $c_\varepsilon \in (\frac{1}{2}, \frac{1}{2} + \varepsilon)$ *and* $\alpha_\varepsilon, \beta_\varepsilon \in \mathbb{Q}$ *such that the following holds: for all large* m, *there is an* $m \times m$ *matrix* $M \in \mathrm{Sym}(\alpha_\varepsilon, \ldots, \alpha_\varepsilon, \beta_\varepsilon, \ldots, \beta_\varepsilon)$ *with* $\mathrm{rank}(M) \leq c_\varepsilon m + O(1)$.

Have you seen a graph like this?

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The sunflower and Hadamard families not only have sets of only two distinct sizes, those sizes are in the proportion 1 : 2.

To improve the trivial bound for such families, we ask:

Question

Is there a bipartite graph with $\sqrt{2}$ *as an eigenvalue with "high multiplicity"?*

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